A First Course in Probability

Tenth Edition

Sheldon Ross

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Pearson

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Pearson

For Rebecca

Contents

Preface

"We see that the theory of probability is at bottom only common sense reduced to calculation; it makes us appreciate with exactitude what reasonable minds feel by a sort of instinct, often without being able to account for it... It is remarkable that this science, which originated in the consideration of games of chance, should have become the most important object of human knowledge.... The most important questions of life are, for the most part, really only problems of probability." So said the famous French mathematician and astronomer (the "Newton of France") Pierre-Simon, Marquis de Laplace. Although many people believe that the famous marquis, who was also one of the great contributors to the development of probability, might have exaggerated somewhat, it is nevertheless true that probability theory has become a tool of fundamental importance to nearly all scientists, engineers, medical practitioners, jurists, and industrialists. In fact, the enlightened individual had learned to ask not "Is it so?" but rather "What is the probability that it is so?"

General Approach and Mathematical Level

This book is intended as an elementary introduction to the theory of probability for students in mathematics, statistics, engineering, and the sciences (including computer science, biology, the social sciences, and management science) who possess the prerequisite knowledge of elementary calculus. It attempts to present not only the mathematics of probability theory, but also, through numerous examples, the many diverse possible applications of this subject.

Content and Course Planning

Chapter 1 presents the basic principles of combinatorial analysis, which are most useful in computing probabilities.

Chapter 2 handles the axioms of probability theory and shows how they can be applied to compute various probabilities of interest.

Chapter 3 deals with the extremely important subjects of conditional probability and independence of events. By a series of examples, we illustrate how conditional probabilities come into play not only when some partial information is available, but also as a tool to enable us to compute probabilities more easily, even when no partial information is present. This extremely important technique of obtaining probabilities by "conditioning" reappears in **Chapter 7**, where we use it to obtain expectations.

The concept of random variables is introduced in **Chapters 4**, **5**, and **6**. Discrete random variables are dealt with in **Chapter 4**, continuous random variables in **Chapter 5**, and jointly distributed random variables in **Chapter 6**. The important concepts of the expected value and the variance of a random variable are introduced in **Chapters 4** and **5**, and these quantities are then determined for many of the common types of random variables.

Additional properties of the expected value are considered in **Chapter 7**. Many examples illustrating the usefulness of the result that the expected value of a sum of random variables is equal to the sum of their expected values are presented. Sections on conditional expectation, including its use in prediction, and on moment-generating functions are contained in this chapter. In addition, the final section introduces the multivariate normal distribution and presents a simple proof concerning the joint distribution of the sample mean and sample variance of a sample from a normal distribution.

Chapter 8 presents the major theoretical results of probability theory. In particular, we prove the strong law of large numbers and the central limit theorem. Our proof of the strong law is a relatively simple one that assumes that the random variables have a finite fourth moment, and our proof of the central limit theorem assumes Levy's continuity theorem. This chapter also presents such probability inequalities as Markov's inequality, Chebyshev's inequality, and Chernoff bounds. The final section of **Chapter 8** gives a bound on the error involved when a probability concerning a sum of independent Bernoulli random variables is approximated by the corresponding probability of a Poisson random variable having the same expected value.

Chapter 9 presents some additional topics, such as Markov chains, the Poisson process, and an introduction to information and coding theory, and **Chapter 10** considers simulation.

As in the previous edition, three sets of exercises are given at the end of each chapter. They are designated as **Problems, Theoretical Exercises**, and **Self-Test Problems and Exercises**. This last set of exercises, for which complete solutions appear in Solutions to Self-Test Problems and Exercises, is designed to help students test their comprehension and study for exams.

Changes for the Tenth Edition

The tenth edition continues the evolution and fine tuning of the text. Aside from a multitude of small changes made to increase the clarity of the text, the new edition includes many new and updated problems, exercises, and text material chosen both for inherent interest and for their use in building student intuition about probability. Illustrative of these goals are Examples 4n of **Chapter 3**, which deals with computing NCAA basketball tournament win probabilities, and Example 5b of **Chapter 4**, which introduces the friendship paradox. There is also new material on the Pareto distribution (introduced in **Section 5.6.5**), on Poisson limit results (in **Section 8.5**), and on the Lorenz curve (in **Section 8.7**).

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Chapter 1 Combinatorial Analysis

Contents

1.1 Introduction

Here is a typical problem of interest involving probability: A communication system is to consist of n seemingly identical antennas that are to be lined up in a linear order.

The resulting system will then be able to receive all incoming signals and will be called *functional* as long as no two consecutive antennas are defective. If it turns out that exactly m of the n antennas are defective, what is the probability that the resulting system will be functional? For instance, in the special case where $n = 4$ and $m = 2$, there are 6 possible system configurations, namely,

$$\begin{matrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{matrix}$$

where 1 means that the antenna is working and 0 that it is defective. Because the resulting system will be functional in the first 3 arrangements and not functional in the remaining 3, it seems reasonable to take $\frac{3}{6} = \frac{1}{2}$ as the desired probability. In the case of general n and m, we could compute the probability that the system is functional in a similar fashion. That is, we could count the number of configurations that result in the system's being functional and then divide by the total number of all possible configurations.

From the preceding discussion, we see that it would be useful to have an effective method for counting the number of ways that things can occur. In fact, many problems in probability theory can be solved simply by counting the number of different ways that a certain event can occur. The mathematical theory of counting is formally known as *combinatorial analysis*.

1.2 The Basic Principle of Counting

The basic principle of counting will be fundamental to all our work. Loosely put, it states that if one experiment can result in any of m possible outcomes and if another experiment can result in any of n possible outcomes, then there are *mn* possible outcomes of the two experiments.

> ### The basic principle of counting
> Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are *mn* possible outcomes of the two experiments.

Proof of the Basic Principle: The basic principle may be proven by enumerating all the possible outcomes of the two experiments; that is,

$$\begin{array}{cccc} (1,1), & (1,2), & \ldots, & (1,n) \\ (2,1), & (2,2), & \ldots, & (2,n) \\ \vdots & & & \\ (m,1), & (m,2), & \ldots, & (m,n) \end{array}$$

where we say that the outcome is (i, j) if experiment 1 results in its ith possible outcome and experiment 2 then results in its jth possible outcome. Hence, the set of possible outcomes consists of m rows, each containing n elements. This proves the result.

Example 2a

A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

Solution

By regarding the choice of the woman as the outcome of the first experiment and the subsequent choice of one of her children as the outcome of the second experiment, we see from the basic principle that there are $10 \times 3 = 30$ possible choices.

When there are more than two experiments to be performed, the basic principle can be generalized.

> ## The generalized basic principle of counting
>
> If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes; and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment; and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment; and if, then there is a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.

Example 2b

A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible?

Solution

We may regard the choice of a subcommittee as the combined outcome of the four separate experiments of choosing a single representative from each of the classes. It then follows from the generalized version of the basic principle that there are $3 \times 4 \times 5 \times 2 = 120$ possible subcommittees.

Example 2c

How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

Solution

By the generalized version of the basic principle, the answer is $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175{,}760{,}000$.

Example 2d

How many functions defined on n points are possible if each functional value is either 0 or 1?

Solution

Let the points be $1,2,\ldots,n$. Since $f(i)$ must be either 0 or 1 for each $i = 1,2,\ldots,n$, it follows that there are 2^n possible functions.

Example 2e

In **Example 2c**, how many license plates would be possible if repetition among letters or numbers were prohibited?

Solution

In this case, there would be $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 78{,}624{,}000$ possible license plates.

1.3 Permutations

How many different ordered arrangements of the letters a, b, and c are possible? By direct enumeration we see that there are 6, namely, *abc, acb, bac, bca, cab*, and *cba*. Each arrangement is known as a *permutation*. Thus, there are 6 possible permutations of a set of 3 objects. This result could also have been obtained from the basic principle, since the first object in the permutation can be any of the 3, the second object in the permutation can then be chosen from any of the remaining 2, and the third object in the permutation is then the remaining 1. Thus, there are $3 \cdot 2 \cdot 1 = 6$ possible permutations.

> Suppose now that we have n objects. Reasoning similar to that we have just used for the 3 letters then shows that there are

> $$n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$$
>
> different permutations of the n objects.

Whereas $n!$ (read as "n factorial") is defined to equal $1\cdot 2\cdots n$ when n is a positive integer, it is convenient to define $0!$ to equal 1.

Example 3a

How many different batting orders are possible for a baseball team consisting of 9 players?

Solution

There are $9! = 362{,}880$ possible batting orders.

Example 3b

A class in probability theory consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.

a. How many different rankings are possible?
b. If the men are ranked just among themselves and the women just among themselves, how many different rankings are possible?

Solution

a. (a) Because each ranking corresponds to a particular ordered arrangement of the 10 people, the answer to this part is $10! = 3{,}628{,}800$.
b. (b) Since there are 6! possible rankings of the men among themselves and 4! possible rankings of the women among themselves, it follows from the basic principle that there are $(6!)(4!) = (720)(24) = 17{,}280$ possible rankings in this case.

Example 3c

Ms. Jones has 10 books that she is going to put on her bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Ms. Jones wants to arrange her books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

Solution

There are 4! 3! 2! 1! arrangements such that the mathematics books are first in line, then the chemistry books, then the history books, and then the language book. Similarly, for each possible ordering of the subjects, there are 4! 3! 2! 1! possible arrangements. Hence, as there are 4! possible orderings of the subjects, the desired answer is $4!\,4!\,3!\,2!\,1! = 6912$.

We shall now determine the number of permutations of a set of n objects when certain of the objects are indistinguishable from one another. To set this situation straight in our minds, consider the following example.

Example 3d

How many different letter arrangements can be formed from the letters $PEPPER$?

Solution

We first note that there are 6! permutations of the letters $P_1E_1P_2P_3E_2R$ when the $3P$'s and the $2E$'s are distinguished from one another. However, consider any one of these permutations for instance, $P_1P_2E_1P_3E_2R$. If we now permute the P's among themselves and the E's among themselves, then the resultant arrangement would still be of the form $PPEPER$. That is, all 3! 2! permutations

$$\begin{array}{ll} P_1P_2E_1P_3E_2R & P_1P_2E_2P_3E_1R \\ P_1P_3E_1P_2E_2R & P_1P_3E_2P_2E_1R \\ P_2P_1E_1P_3E_2R & P_2P_1E_2P_3E_1R \\ P_2P_3E_1P_1E_2R & P_2P_3E_2P_1E_1R \\ P_3P_1E_1P_2E_2R & P_3P_1E_2P_2E_1R \\ P_3P_2E_1P_1E_2R & P_3P_2E_2P_1E_1R \end{array}$$

are of the form $PPEPER$. Hence, there are $6!/(3!\,2!) = 60$ possible letter arrangements of the letters $PEPPER$.

In general, the same reasoning as that used in **Example 3d** shows that there are

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

different permutations of n objects, of which n_1 are alike, n_2 are alike, $\ldots, n_r$ are alike.

Example 3e

A chess tournament has 10 competitors, of which 4 are Russian, 3 are from the United States, 2 are from Great Britain, and 1 is from Brazil. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?

Solution

There are

$$\frac{10!}{4!\,3!\,2!\,1!} = 12{,}600$$

possible outcomes.

Example 3f

How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags if all flags of the same color are identical?

Solution

There are

$$\frac{9!}{4!\,3!\,2!} = 1260$$

different signals.

1.4 Combinations

We are often interested in determining the number of different groups of r objects that could be formed from a total of n objects. For instance, how many different groups of 3 could be selected from the 5 items A, B, C, D, and E? To answer this question, reason as follows: Since there are 5 ways to select the initial item, 4 ways to then select the next item, and 3 ways to select the final item, there are thus $5 \cdot 4 \cdot 3$ ways of selecting the group of 3 when the order in which the items are selected is relevant. However, since every group of 3–say, the group consisting of items A, B, and C will be counted 6 times (that is, all of the permutations *ABC, ACB, BAC, BCA, CAB*, and *CBA* will be counted when the order of selection is relevant), it follows that the total number of groups that can be formed is

$$\frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$$

In general, as $n(n-1)\cdots(n-r+1)$ represents the number of different ways that a group of r items could be selected from n items when the order of selection is relevant, and as each group of r items will be counted $r!$ times in this count, it follows that the number of different groups of r items that could be formed from a set of n

items is

$$\frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!\,r!}$$

Notation and terminology

We define $\binom{n}{r}$, for $r \leq n$, by

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

and say that $\binom{n}{r}$ (read as “n choose r“) represents the number of possible combinations of n objects taken r at a time.

Thus, $\binom{n}{r}$ represents the number of different groups of size r that could be selected from a set of n objects when the order of selection is not considered relevant.

Equivalently, $\binom{n}{r}$ is the number of subsets of size r that can be chosen from a set of size n. Using that $0! = 1$, note that $\binom{n}{n} = \binom{n}{0} = \frac{n!}{0!n!} = 1,$ which is consistent with the preceding interpretation because in a set of size n there is exactly 1 subset of size n (namely, the entire set), and exactly one subset of size 0 (namely the empty set). A useful convention is to define $\binom{n}{r}$ equal to 0 when either $r > n$ or $r < 0$.

Example 4a

A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

Solution

There are $\binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140$ possible committees.

Example 4b

From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed? What if 2 of the men are feuding and refuse to serve on the committee together?

Solution

As there are $\binom{5}{2}$ possible groups of 2 women, and $\binom{7}{3}$ possible groups of 3 men, it follows from the basic principle that there are $\binom{5}{2}\binom{7}{3} = \frac{5 \cdot 4}{2 \cdot 1}$. $\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 350$ possible committees consisting of 2 women and 3 men.

Now suppose that 2 of the men refuse to serve together. Because a total of $\binom{2}{2}\binom{5}{1} = 5$ out of the $\binom{7}{3} = 35$ possible groups of 3 men contain both of the feuding men, it follows that there are $35 - 5 = 30$ groups that do not contain both of the feuding men. Because there are still $\binom{5}{2} = 10$ ways to choose the 2 women, there are $30 \cdot 10 = 300$ possible committees in this case.

Example 4c

Consider a set of n antennas of which m are defective and $n - m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

Solution

Imagine that the $n - m$ functional antennas are lined up among themselves. Now, if no two defectives are to be consecutive, then the spaces between the functional antennas must each contain at most one defective antenna. That is, in the $n - m + 1$ possible positions–represented in **Figure 1.1** by carets–between the $n - m$ functional antennas, we must select m of these in which to put the defective antennas. Hence, there are $\binom{n-m+1}{m}$ possible orderings in which there is at least one functional antenna between any two defective ones.

Figure 1.1 No consecutive defectives.

The figure shows No consecutive defectives

A useful combinatorial identity, known as *Pascal's identity*, is

(4.1)

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \quad 1 \leq r \leq n$$

Equation (4.1) may be proved analytically or by the following combinatorial argument: Consider a group of n objects, and fix attention on some particular one of these objects–call it object 1. Now, there are $\binom{n-1}{r-1}$ groups of size r that contain object 1 (since each such group is formed by selecting $r-1$ from the remaining $n-1$ objects). Also, there are $\binom{n-1}{r}$ groups of size r that do not contain object 1. As there is a total of $\binom{n}{r}$ groups of size r, **Equation (4.1)** follows.

The values $\binom{n}{r}$ are often referred to as *binomial coefficients* because of their prominence in the binomial theorem.

The binomial theorem

(4.2)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

We shall present two proofs of the binomial theorem. The first is a proof by mathematical induction, and the second is a proof based on combinatorial considerations.

Proof of the Binomial Theorem by Induction: When $n = 1$, **Equation (4.2)** reduces to

$$x + y = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0 = y + x$$

Assume **Equation (4.2)** for $n-1$. Now,

$$\begin{aligned}
(x+y)^n &= (x+y)(x+y)^{n-1} \\
&= (x+y)\sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-1-k} \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}x^{k+1}y^{n-1-k} + \sum_{k=0}^{n-1}\binom{n-1}{k}x^k y^{n-k}
\end{aligned}$$

Letting $i = k+1$ in the first sum and $i = k$ in the second sum, we find that

$$
\begin{aligned}
(x+y)^n &= \sum_{i=1}^{n} \binom{n-1}{i-1} x^i y^{n-i} + \sum_{i=0}^{n-1} \binom{n-1}{i} x^i y^{n-i} \\
&= \sum_{i=1}^{n-1} \binom{n-1}{i-1} x^i y^{n-i} + x^n + y^n + \sum_{i=1}^{n-1} \binom{n-1}{i} x^i y^{n-i} \\
&= x^n + \sum_{i=1}^{n-1} \left[\binom{n-1}{i-1} + \binom{n-1}{i} \right] x^i y^{n-i} + y^n \\
&= x^n + \sum_{i=1}^{n-1} \binom{n}{i} x^i y^{n-i} + y^n \\
&= \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}
\end{aligned}
$$

where the next-to-last equality follows by **Equation (4.1)** . By induction, the theorem is now proved.

Combinatorial Proof of the Binomial Theorem: Consider the product

$$(x_1 + y_1)(x_2 + y_2)\cdots(x_n + y_n)$$

Its expansion consists of the sum of 2^n terms, each term being the product of n factors. Furthermore, each of the 2^n terms in the sum will contain as a factor either x_i or y_i for each $i = 1, 2, \ldots, n$. For example,

$$(x_1 + y_1)(x_2 + y_2) = x_1x_2 + x_1y_2 + y_1x_2 + y_1y_2$$

Now, how many of the 2^n terms in the sum will have k of the x_i's and $(n-k)$ of the y_i's as factors? As each term consisting of k of the x_i's and $(n-k)$ of the y_i's corresponds to a choice of a group of k from the n values $x_1, x_2 \ldots, x_n$, there are $\binom{n}{k}$ such terms. Thus, letting $x_i = x, y_i = y, i = 1, \ldots, n$, we see that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example 4d

Expand $(x+y)^3$.

Solution

$$\begin{aligned}(x+y)^3 &= \binom{3}{0}x^0y^3 + \binom{3}{1}x^1y^2 + \binom{3}{2}x^2y^1 + \binom{3}{3}x^3y^0 \\ &= y^3 + 3xy^2 + 3x^2y + x^3\end{aligned}$$

Example 4e

How many subsets are there of a set consisting of n elements?

Solution

Since there are $\binom{n}{k}$ subsets of size k, the desired answer is

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

This result could also have been obtained by assigning either the number 0 or the number 1 to each element in the set. To each assignment of numbers, there corresponds, in a one-to-one fashion, a subset, namely, that subset consisting of all elements that were assigned the value 1. As there are 2^n possible assignments, the result follows.

Note that we have included the set consisting of 0 elements (that is, the null set) as a subset of the original set. Hence, the number of subsets that contain at least 1 element is $2^n - 1$.

1.5 Multinomial Coefficients

In this section, we consider the following problem: A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$. How many different divisions are possible? To answer this question, we note that there are $\binom{n}{n_1}$ possible choices for the first group; for each choice of the first group, there are $\binom{n-n_1}{n_2}$ possible choices for the second group; for each choice of the first two groups, there are $\binom{n-n_1-n_2}{n_3}$ possible choices for the third group; and so on. It then follows from the generalized version of the basic counting principle that there are

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$$
$$=\frac{n!}{(n-n_1)!\,n_1!}\,\frac{(n-n_1)!}{(n-n_1-n_2)!\,n_2!}\cdots\frac{(n-n_1-n_2-\cdots-n_{r-1})!}{0!\,n_r!}$$
$$=\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

possible divisions.

Another way to see this result is to consider the n values $1,1,\ldots,1,2,\ldots,2,\ldots,$ $r,\ldots,r$, where i appears n_i times, for $i=1,\ldots,r$. Every permutation of these values corresponds to a division of the n items into the r groups in the following manner: Let the permutation $i_1,i_2,\ldots,i_n$ correspond to assigning item 1 to group i_1, item 2 to group i_2, and so on. For instance, if $n=8$ and if $n_1=4, n_2=3,$ and $n_3=1$, then the permutation $1,1,2,3,2,1,2,1$ corresponds to assigning items $1,2,6,8$ to the first group, items $3,5,7$ to the second group, and item 4 to the third group. Because every permutation yields a division of the items and every possible division results from some permutation, it follows that the number of divisions of n items into r distinct groups of sizes $n_1,n_2,\ldots,n_r$ is the same as the number of permutations of n items of which n_1 are alike, and n_2 are alike, $\ldots$, and n_r are alike, which was shown in **Section 1.3** to equal $\frac{n!}{n_1!n_2!\cdots n_r!}$.

Notation

If $n_1+n_2+\cdots+n_r=n$, we define $\binom{n}{n_1,n_2,\ldots,n_r}$ by

$$\binom{n}{n_1,n_2,\ldots,n_r}=\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

Thus, $\binom{n}{n_1,n_2,\ldots,n_r}$ represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1,n_2,\ldots,n_r$.

Example 5a

A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the 3 groups are possible?

Solution

There are $\frac{10!}{5!\,2!\,3!} = 2520$ possible divisions.

Example 5b

Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible?

Solution

There are $\frac{10!}{5!\,5!} = 252$ possible divisions.

Example 5c

In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

Solution

Note that this example is different from **Example 5b** because now the order of the two teams is irrelevant. That is, there is no A or B team, but just a division consisting of 2 groups of 5 each. Hence, the desired answer is

$$\frac{10!/(5!\,5!)}{2!} = 126$$

The proof of the following theorem, which generalizes the binomial theorem, is left as an exercise.

The multinomial theorem

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(n_1, \ldots, n_r): \\ n_1 + \cdots + n_r = n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

That is, the sum is over all nonnegative integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$.

The numbers $\binom{n}{n_1, n_2, \ldots, n_r}$ are known as *multinomial coefficients*.

Example 5d

In the first round of a knockout tournament involving $n = 2^m$ players, the n

players are divided into $n/2$ pairs, with each of these pairs then playing a game. The losers of the games are eliminated while the winners go on to the next round, where the process is repeated until only a single player remains. Suppose we have a knockout tournament of 8 players.

a. How many possible outcomes are there for the initial round? (For instance, one outcome is that 1 beats 2, 3 beats 4, 5 beats 6, and 7 beats 8.)
b. How many outcomes of the tournament are possible, where an outcome gives complete information for all rounds?

Solution

One way to determine the number of possible outcomes for the initial round is to first determine the number of possible pairings for that round. To do so, note that the number of ways to divide the 8 players into a *first* pair, a *second* pair, a *third* pair, and a *fourth* pair is $\binom{8}{2,\ 2,\ 2,\ 2} = \frac{8!}{2^4}$. Thus, the number of possible pairings when there is no ordering of the 4 pairs is $\frac{8!}{2^4 4!}$. For each such pairing, there are 2 possible choices from each pair as to the winner of that game, showing that there are $\frac{8!2^4}{2^4 4!} = \frac{8!}{4!}$ possible results of round 1. [Another way to see this is to note that there are $\binom{8}{4}$ possible choices of the 4 winners and, for each such choice, there are $4!$ ways to pair the 4 winners with the 4 losers, showing that there are $4!\binom{8}{4} = \frac{8!}{4!}$ possible results for the first round.]

Similarly, for each result of round 1, there are $\frac{4!}{2!}$ possible outcomes of round 2, and for each of the outcomes of the first two rounds, there are $\frac{2!}{1!}$ possible outcomes of round 3. Consequently, by the generalized basic principle of counting, there are $\frac{8!}{4!}\frac{4!}{2!}\frac{2!}{1!} = 8!$ possible outcomes of the tournament. Indeed, the same argument can be used to show that a knockout tournament of $n = 2^m$ players has $n!$ possible outcomes.

Knowing the preceding result, it is not difficult to come up with a more direct argument by showing that there is a one-to-one correspondence between the set of possible tournament results and the set of permutations of $1, \ldots, n$. To obtain such a correspondence, rank the players as follows for any tournament result: Give the tournament winner rank 1, and give the final-round loser rank 2. For the two players who lost in the next-to-last round, give rank 3 to the one who lost to the player ranked 1 and give rank 4 to the one who lost to the player ranked 2. For the four players who lost in the second-to-last round, give rank 5 to the one who lost to player ranked 1, rank 6 to the one who lost to the player ranked 2, rank 7 to the one who lost to the player ranked 3, and rank 8 to the one who lost to the player ranked 4. Continuing on in this manner gives a rank to each player. (A more succinct description is to give the winner of the tournament rank 1 and let the rank of a player who lost in a round having 2^k matches be 2^k plus the rank of the player who beat him, for $k = 0, \ldots, m-1$.) In this manner, the result of the tournament can be represented by a permutation $i_1, i_2, \ldots, i_n$, where i_j is the player who was given rank j. Because different tournament results give rise to different permutations, and because there is a tournament result for each permutation, it follows that there are the same number of possible tournament results as there are permutations of $1, \ldots, n$.

Example 5e

$$\begin{aligned}(x_1+x_2+x_3)^2 &= \binom{2}{2,\,0,\,0}x_1^2x_2^0x_3^0 + \binom{2}{0,\,2,\,0}x_1^0x_2^2x_3^0 \\ &\quad + \binom{2}{0,\,0,\,2}x_1^0x_2^0x_3^2 + \binom{2}{1,\,1,\,0}x_1^1x_2^1x_3^0 \\ &\quad + \binom{2}{1,\,0,\,1}x_1^1x_2^0x_3^1 + \binom{2}{0,\,1,\,1}x_1^0x_2^1x_3^1 \\ &= x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3\end{aligned}$$

* 1.6 The Number of Integer Solutions of Equations

* Asterisks denote material that is optional.

An individual has gone fishing at Lake Ticonderoga, which contains four types of fish: lake trout, catfish, bass, and bluefish. If we take the result of the fishing trip to be the numbers of each type of fish caught, let us determine the number of possible outcomes when a total of 10 fish are caught. To do so, note that we can denote the outcome of the fishing trip by the vector (x_1, x_2, x_3, x_4) where x_1 is the number of trout that are caught, x_2 is the number of catfish, x_3 is the number of bass, and x_4 is the number of bluefish. Thus, the number of possible outcomes when a total of 10 fish are caught is the number of nonnegative integer vectors (x_1, x_2, x_3, x_4) that sum to 10.

More generally, if we supposed there were r types of fish and that a total of n were caught, then the number of possible outcomes would be the number of nonnegative integer-valued vectors $x_1, \dots, x_r$ such that

(6.1)

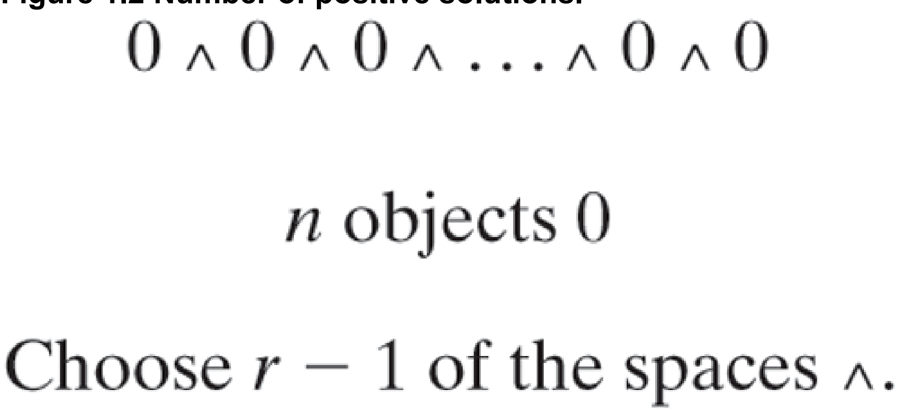
$$x_1 + x_2 + \dots + x_r = n$$

To compute this number, let us start by considering the number of positive integer-valued vectors $x_1, \dots, x_r$ that satisfy the preceding. To determine this number, suppose that we have n consecutive zeroes lined up in a row:

$$0\,0\,0\,\dots\,0\,0$$

Note that any selection of $r-1$ of the $n-1$ spaces between adjacent zeroes (see **Figure 1.2**) corresponds to a positive solution of **6.1** by letting x_1 be the number of zeroes before the first chosen space, x_2 be the number of zeroes between the first and second chosen space, $\dots$, and x_n being the number of zeroes following the last chosen space.

Figure 1.2 Number of positive solutions.



For instance, if we have $n = 8$ and $r = 3$, then (with the choices represented by dots) the choice

$$0.0\,0\,0\,0.0\,0\,0$$

corresponds to the solution $x_1 = 1, x_2 = 4, x_3 = 3$. As positive solutions of (6.1) correspond, in a one-to-one fashion, to choices of $r-1$ of the adjacent spaces, it follows that the number of differerent positive solutions is equal to the number of different selections of $r-1$ of the $n-1$ adjacent spaces. Consequently, we have the following proposition.

Proposition 6.1